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Optimizing Geometric Constraints: A Systematic Review on the Impact of Linear Curvature Regularization in Variational Joint Segmentation-Registration Models for Medical Images

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ABSTRACT

Biomedical imaging has played a vital role in disease detection, tracking, and treatment. These methods are essential for patient care and health benefits. A synthesis of the literature is helpful to find out limitations such as difficulty handling images with complex background structures and intensity inhomogeneity. The primary aim of this review is to tackle these inadequacies. Then, to identify gaps for future studies. A main goal is to review a joint segmentation–registration model that highlights computational and methodological challenges. Also, to identify obstacles to the clinical translation of the new joint segmentation and registration (NJSR) model. A systematic literature review was carried out to specify relevant studies in medical image analysis. Searches were conducted in IEEE Xplore and Google Scholar. Key search terms incorporated "medical image analysis" and "joint segmentation and registration." The review incorporated models across anatomical structures (e.g., brain magnetic resonance imaging (MRI), hand X-ray), and comparative and narrative reviews of main foundational studies. It excluded the non-joint studies. The literature studies show that separate approaches as compared to joint segmentation and registration frameworks, do not show much significant elevation in performance. The comparative and narrative analyses focused on sum of squared differences (SSD) metrics and the deformation vector field. All in all, the previous studies show that joint frameworks can cope with the reliability, accuracy, and robustness of disease detection and tracking process in biomedical image analysis. All in all, NJSR has demonstrated better performance in convergence speed and registration smoothness compared to Guyder Vese joint segmentation and registration (GV-JSR) model, while NJSR also having the limitations in segmentation accuracy for complex real-world data (Experiment 2 and 4) that is in certain complex experiments the F measure is observed to be the lower in value (that is lower than 0.5). This slight sacrifice in the segmentation accuracy (F-measure) is an acceptable trade-off within the context of pure registration task focused on biomechanical plausibility (as shown by lower E score). In regions where the main objective is to derive a physically meaningful motion field for non-rigid compensation—rather than accurate segmentation borders—this enforcement shifts to achieving a smoother, more reliable deformation field. However, it is acknowledged that for clinical segmentation tasks (e.g., tumor volume definition), this level of precision would be insufficient and require further refinement of the data fidelity term. Moreover, the models having NJSR as a base paper having multi modal frameworks are also come into comparison with each other.

INDEX TERMS: Modalities, Variational Frameworks, Levelset, Linear Curvature Regularization, Sum of Squared Difference, Mutual Information, Bhattacharyya Distance, Conditional Mutual Information, Nonlinear Elasticity, Multigrid Methods, and Diffeomorphic Registration.

I. INTRODUCTION

Medical imaging is a connection between two broader fields, biomedical engineering and computer vision. This significant field allows technology and clinical imaging to achieve accurate disease detection and tracking. Advances in imaging modalities—such as ultrasound (US), magnetic resonance imaging (MRI), positron emission tomography (PET), and computed tomography (CT)—have produced various imaging datasets, leading to a necessity of robust joint models efficient in dealing with multimodal registration. This is a task that existent methods are unable to deal with.

These image datasets enables clinical judgement and improve disease detection and tracking accuracy. Thus, leading to improved healthcare outcomes.

Manual analysis is time-consuming; automated methods can greatly reduce clinician workload. In turn, there is a need for joint models that perform the two fundamental tasks of segmentation and registration. Image registration aligns a template image to a reference image, supporting tracking and comparison across images. The reference image remains fixed, while the template image is deformed so that it corresponds to the reference image. In brief, registration geometrically transforms one image to align another. Registration can be mono-modal (single imaging modality) or multimodal (multiple imaging modalities) [1]. Applications of image alignment include computational anatomy [1], [2], computer aided diagnosis [1], [3], radiation therapy [1], [4], treatment verification [1], [5], CCTV (Closed-Circuit Television) analysis [1], [6] and remote sensing [1], [7]. Segmentation is the process of dividing an image into meaningful regions and setting apart foreground from background. It involves deriving objects from their surrounding background [8]–[10]. Conventional segmentation techniques include histogram/thresholding methods, active contours, edge detection [11]–[13], and region growing [14], [15] and region-based segmentation [16] are different techniques used for segmentation of images. Segmentation often acts as a pre-processing step for registration, since detecting regions of interest can facilitate image matching. Traditional sequential approaches are inefficient and prone to error propagation, motivating joint optimization frameworks.

Although segmentation and registration have traditionally been performed separately, researchers increasingly recognize their interdependence and propose joint models with a single objective function. This assists in improving the accuracy of both the tasks together and would avoid error propagation. Its importance increasing beyond algorithmic efficiency to important applications in medical domain. Recent progressive studies in deep learning, computer vision, and applied mathematics have promoted rapid increase in joint segmentation–registration research. While many studies based on either segmentation or registration. This leads to a need of joint frameworks to enhance algorithmic technicality and consistency. The NJSR model includes the linear curvature term as its regularization term. It leads to implement the structural consistency. It is one of the factor retaining relevance in the era of deep learning. The regularizer used here compels smoothness in transformation. The diffeomorphism (non-folding) is compelled by positive Jacobian determinant constraint. This review article highlights the lack of advance analyses of comparing NJSR model with the earlier GV-JSR framework [17]. Moreover, it also involved the multimodal joint models which is having NJSR as a base paper. Furthermore, medical applications need valid and efficient techniques; therefore, stable, reliable, and valid evaluation is crucial. Recent advancements in joint image segmentation and registration have thrown light on combining deep learning (DL) and variational methods to enhance the performance across various datasets. Such as, unified multimodal formulations with their implicit neural network forms have been demonstrated in advance studies [18]. Also, multi-scale attention integration techniques have shown improved accuracy in deformable registration fields [19]. Furthermore, recent reviews emphasize evolving research directions and complex tasks in DL based registration areas [20].

A new synthesis is necessary making it difficult to identify the most effective approaches.

This review summarizes and compares existing joint segmentation–registration models in medical image analysis. To achieve this aim, the review is driven by the following objectives: (1) identify joint segmentation–registration models proposed in the literature, (2) compare their performance and evaluation metrics, (3) examine strengths and limitations, (4) identify research gaps, and (5) assess the overall progress and scope of the field.

Based on these objectives the following are the research questions:

RQ1: What evaluation metrics and methods used in GV-JSR, NJSR and the other multimodal frameworks?

RQ2: How does the NJSR model compare to the GV-JSR model regarding accuracy and consistency?

RQ3: What are the limitations of the existent NJSR and the other latter joint models?

RQ4: What specific algorithmic modifications (e.g., parameter selection numerical scheme) are required to transition NJSR to 3D multimodal clinical use?

II. RELATED WORK

The Mumford–Shah model [21] formulates image segmentation as the problem of approximating images with piecewise smooth functions while preserving significant edges. Its primary objective is to develop a variational framework that efficiently represents regions and boundaries in images. To let this task achieve, the authors used optimization techniques and variational calculus. The functional includes ensurity in local smoothness, data fidelity, perimeter. This gives meaningful image detection with sharp edges. The authors further explained this model is a classical foundation for many segmentation models. At last, it can be extended to global optimization problem. However, the non-convex computational stability of the optimization give ways to an effective modelling and scalable executions.

The Chan–Vese model deals with segmentation when object boundaries are not defined by strong gradients. It shows its strength compared to classical gradient-based snake models. The primary aim is to propose a variational, region-based model that improves segmentation in noisy or low-contrast images. To achieve this, the authors employed a piecewise constant approximation of the Mumford–Shah functional. This implies an energy minimization framework that is based on region-based detection and level-set methods for contour evolution.

The authors further discuss that these results extend the variational approach of Mumford–Shah in a simplified and computationally practical form, linking mathematical theory to more effective applications in medical and natural image segmentation. In conclusion, the paper highlights that region-based active contours provide a powerful alternative to edge-based methods and suggests further work in extending the model to multi-phase segmentation, textured images, and more efficient numerical implementations. However, limitations include the piecewise-constant assumption, sensitivity to parameter selection, and primary focus on binary segmentation. The segmentation term in the NJSR functional is a direct adaptation of Chan-Vese energy functional.

The GV-JSR model integrates segmentation and registration within a single framework to avoid errors introduced by sequential two-stage approaches. The fundamental aim of the study is to design a unified variational model that simultaneously performs segmentation and registration, while enforcing smoothness of the deformation field through nonlinear elasticity, thereby improving both robustness and accuracy. To handle this, the authors used a variational formulation that couples an active contour segmentation model with a nonlinear elasticity regularization for the deformation field. It has solved the resulting system using partial differential equations based numerical optimization and level-set methods.

The study indicates that the model has used non-linear elasticity (NLE) term as a regularizer. It acts as a rigor to enforce smoothness than previous models. It also ensures reliable deformation than previous models. It indicates robustness than previous methods in maintaining image topology. It may include future work to deal with multimodal registration, stability of binary segmentation and enhance algorithmic efficiency.

$$\min\ J(k_1, k_2, p(a)) = \gamma_1 \int |I(a) - k_1|^2 H\varepsilon(\phi_0(a)(a + p(a)))\, da + \gamma_2 \int |I(a) - k_2|^2(1 - H\varepsilon(\phi_0(a)(a + p(a))))\, da + AS^{NLE}(b) + AB\|b - \nabla(p(a))\|^2 \quad (1)$$

k_1 and k_2 are the average intensities inside and outside the curve Γ in the reference image which is represented by the zero level line. The regularization term in GV-JSR model made use of b as a dual variable for $\nabla p(a)$. It is non-linear in nature. This term intrinsically regularizes deformation more efficiently than linear terms. It also ensures p to remain consistent, hence, gaining supremacy over the NJSR model having linear regularizer term. Besides its advantages it has some shortcomings as well. It does not adequately enforce smoothness over the deformation field. The primary aim of this study is to form such a joint model having linear curvature smoother term as a regularizer. It proved to provide a reliable, precise and algorithmically efficient method. To tackle this, the authors made use of region based segmentation model known as Chan Vese (CV) model with linear curvature smoothing term for well regularizing the deformation field.

It solved the coupled PDE system using efficient numerical optimization techniques. The study shows that the proposed model yields smoother deformation fields and more accurate segmentation results compared to existing joint models. It shows the significance of using linear curvature regularization. It is more efficient and less robust than non-linear terms for extreme deformation. The authors extended that these results widely improved the mathematical consistency of joint frameworks and show improved convergence behavior relative to nonlinear smoothing frameworks.

In conclusion, the paper highlights that the linear curvature–based joint model is an effective advancement in segmentation–registration methods but along with some limitations it does not completely impose a diffeomorphic mapping, that is, true physical validity requires positive Jacobian value, that ensures $\det(\nabla p) > 0$ almost everywhere. It can not be said that linear curvature (LC) guarantees mathematically against folding but smoothness. It may in future be extended the framework to multimodal images, larger 3D datasets, and clinical validation. However, the study is limited by its reliance on binary segmentation, potential computational expense in large-scale problems, and testing on relatively simple datasets, which indicates areas for further research.

Rada et al. define the functional as:

$$\min\ J(k_1, k_2, p(a)) = \gamma_1 \int |I(a) - k_1|^2 H\varepsilon(\phi_0(a)(a + p(a)))\, da + \gamma_2 \int |I(a) - k_2|^2(1 - H\varepsilon(\phi_0(a)(a + p(a))))\, da + D^{SSDH}(M, I, \phi_0(a)(x), p(a)) + AS^{LC}(p) \quad (2)$$

$$k_1 = \int I(a)\, H\varepsilon(\phi_0(a)(a + p(a)))\, da \Big/ \int H\varepsilon(\phi_0(a)(a + p(a)))\, da \quad (3)$$

$$k_2 = \int I(a)(1 - H\varepsilon(\phi_0(a)(a + p(a))))\, da \Big/ \int (1 - H\varepsilon(\phi_0(a)(a + p(a))))\, da \quad (4)$$

where I(a) represents reference image and M represents template image, γ_1, γ_2, γ_3 and A are weights, first two terms are segmentation terms using Chan-Vese averages and D^{SSDH} (sum of squared differences with Heaviside function (SSDH)) is a weighted L2 norm of the difference in the intensity value between the two images and S^{LC} p(a) is the linear curvature term to regularize the deformation field. The regmentation model [22] solved the problem of absence of multimodal joint frameworks and extended NJSR model into multimodality. The main objectives of this model includes work on multimodal medical images, used CV model for segmentation, mutual information (MI) for registration and used linear curvature (LC) as a regularizer to keep the deformation field smooth.

It has shown many advantages to medical field that incorporates, first multimodal joint framework introduced, it has also enhanced the need of joint models rather than separate segmentation and registration models. It deals with both shape changes and alignment together in a one unified model. Alongside its advantages, it also shares some limitations that includes MI which is sensitive to noise, not good performance while having intensity inhomogeneity observed in images, not good for images having objects with weak or blurry boundaries and it get stuck in local minima. It concludes that, the regmentation model has played a role of a foundational work for multimodal medical images but limited due to severe noise and intensity inhomogeneity.

The functional is defined as:

$$\min\ J[k_1, k_2, p] = k_1 \int |I(a) - k_1|^2 H\varepsilon(\phi_0(a + p(a)))\, da + k_2 \int |I(a) - k_2|^2(1 - H\varepsilon(\phi_0(a)(a + p(a))))\, da + cS^{LC}[p] + \ell D^{H}_{MI}(M, I, \phi_0, p) \quad (5)$$

$$S^{LC}[p] = \int (|\nabla p_1(a)|^2 + |\nabla p_2(a)|^2)\, da \quad (6)$$

and Hε is the regularized Heaviside. The Badshah et al. model [23] that is more robust than regmentation model and filled the gap that mutual information (MI) failed in noisy and biased dataset. It incorporates active contour model for segmentation and MI was replaced by Bhattacharyya distance for registration. It is having vital contribution that it works in severe noise and intensity inhomogeneity in multimodal medical dataset. It has still faced some limitations that it did not work well with weak edges, segmentation not fully localized and some parameters dependency. It would be concluded as this model has improved the efficiency of regmentation model by replacing MI by Bhattacharyya distance term and lead to better stability and performance. Its framework involved the following:

$$\min J[k_{in}, k_{out}, p(a)] = m_1 \int |I_0(a) - k_{in}|^2 H\varepsilon(\phi_0(a + p(a)))\, da + m_2 \int |I_0(a) - k_{out}|^2 (1 - H\varepsilon(\phi_0(a + p(a))))\, da - m_3 \iint \sqrt{(\rho_M(a_0)\, \rho_{I_0}(b_0)\, \rho_{M,I_0}(a_0, b_0))}\, d(a_0, b_0) + \alpha \int (|\nabla p_1(a)|^2 + |\nabla p_2(a)|^2)\, da \quad (7)$$

The latter Badshah et al. model [24] addressed the gaps of regmentation and 2022 Badshah et al. model that they were facing that is weak boundaries, blurry areas, severe noise and global similarity weakness. It has introduced in multimodal framework the local binary fitting (LBF) to handle local region information and conditional mutual information (CMI) to deal with the weak edges. Its vital aim is to design a highly robust and stable joint model that would be able to deal with multimodality, noise and inhomogeneity. Moreover, it is capable of dealing with weak boundaries, does not require reinitialization and utilizes both global and local information. It has advantages of performing better than the previous two multimodal frameworks and showed strong quantitative improvement. It has some limitations as well that it has high computational cost and it would more complex mathematically and harder to implement in MATLAB/Python. It would be concluded as it has proved to be the most effective model among the previous joint models by showing improved quantitative performance among them. Its energy functional is:

$$\min J(\phi_0, \hat{k}_1, \hat{k}_2, p) = v_1 \iint G\sigma(a_2 - a_1)|I_0(a_1) - \hat{k}_1|^2 H\varepsilon(\phi_0(a_1 + p(a_1)))\, da_2\, da_1 + v_2 \iint G\sigma(a_2 - a_1)|I_0(a_1) - \hat{k}_2|^2 (1 - H\varepsilon(\phi_0(a_1 + p(a_1))))\, da_2\, da_1 + v_3 D^{H}{}_{CMI}(M, I_0, \phi_0, p) + \gamma S^{LC}[p] \quad (8)$$

III. METHODOLOGY

Information sources: Searches were carried out in Institute of electrical and electronics engineers (IEEE) Xplore and Google Scholar on 12/03/2025.

A systematic literature review was conducted to trace the evolution of joint segmentation-registration models from the classical Mumford-Shah/Chan-Vese approaches [11], [21] to the modern NJSR model [25] and its multimodal extensions [22]–[24]. Initial papers were identified using core keywords (e.g., 'joint segmentation-registration,' 'linear curvature regularization') in databases such as Google Scholar and Scopus. Subsequent relevant literature was identified through forward and backward citation searching of these foundational papers, prioritizing works that mathematically shows the impact of the linear curvature term on the optimization problem scenario.

Other search methods: Only backward snowballing was applied to track conceptual and foundational work (e.g., the Chan–Vese model and the GV-JSR model). It has helped connecting NJSR model with the classical and state of art model. Forward snowballing was also used because the objective was to trace not only foundational and conceptual models leading to NJSR but also to survey subsequent extensions.

Eligibility criteria: Studies must involve medical image data and address joint segmentation–registration models. Intervention/exposure studies reviewed include extensions of the GV-JSR, NJSR, Regmentation, Badshah et al. 2022 and Badshah et al. 2024 models. Comparison studies have compared NJSR to the GV-JSR model and Regmentation model to each of Badshah et al. models.

Outcomes studies are able to cope with segmentation of exterior boundaries of image and to register interior structures efficiently. Study design studies has used methodological and experimental research for the valid and reliable outcomes. Other criteria incorporates full text availability, published in English between 2001-2024, includes relevant grey literature [1].

Inclusion criteria incorporated, studies involved joint segmentation and registration models, published joint models from 2009 to 2024, peer-reviewed, report quantitative evaluation metrics. Exclusion criteria, involved review articles, abstract only, non-medical image analysis. For each selected NJSR study we extracted: authors, year, study type (e.g., experimental, methodological), datasets type (MRI/CT), image size (e.g., 128 × 128), optimization methods, performance metrics (e.g., SSD), main findings, and limitations. Study quality was evaluated using the F-measure for segmentation accuracy. Moreover, its limitations and algorithmic modifications are suggested to fill and improve in future work. The studies have evaluated the risk of quality assessment by the F-measure.

Two studies were included in the results after reviewing 100 articles. The selected work published in 2001-2024. It mainly involved experimental and methodological studies. The results were organized around three main aspects that are, model efficiency, joint model accuracy, computational optimization.

Model Efficiency: The NJSR model improved convergence and this is due to use of linear curvature model and lead to better efficiency than GV-JSR model. The joint model has improved its segmentation accuracy than state of art model and achieved accurate region boundaries and better deformation.

For computational optimization, the studies has used multilevel approach and reduced E-score value of NJSR model than GV-JSR model indicating its optimized computationally as compare to GV-JSR model. Using a hierarchy of discretization with a pyramid of grids, multigrid methods (MG), also referred to as multilevel methods, have been demonstrated to be an effective solver for both linear and nonlinear elliptic PDEs. The NJSR model has used this method to handle large deformation, starting from coarser level using rough

Table 1: Summary of Literature Review

Research Paper Reference Number	Problem solved by authors	Research Framework	Techniques	Research Gaps	Conclusion / Future work
[21]	Approximating images with piecewise smooth functions allowing sharp discontinuities.	Variational segmentation model minimizing data fidelity, smoothness, and edge penalties.	PDE-based optimization for the Mumford–Shah functional.	Direct computational implementation is not feasible.	Foundation for modern segmentation and denoising methods.
[11]	Overcomes failure of gradient-based snakes on weak/noisy edges.	Region-based Chan–Vese active contour model.	Level-set methods, PDEs, finite differences.	Not effective for images with intensity inhomogeneity.	Widely used variational segmentation method not dependent on gradients.
[17]	Performs segmentation and registration jointly for medical imaging.	Unified variational model with nonlinear elasticity smoother.	Level-set methods, B-splines, BFGS optimization.	Insufficient deformation of interior foreground structures; limited to single-object images.	Achieves smooth, accurate joint segmentation and registration.
[25]	Removes irregular deformation and pre-registration dependence.	Unified registration–segmentation variational framework.	B-spline deformation fields, PDE solvers, quasi-Newton optimization.	It would not work well if complexities in background structure.	Provides smoother, more robust transformations.
[22]	The authors target the difficulty of performing segmentation and registration jointly but in multimodal images.	Unified registration–segmentation variational framework with mutual information.	Multi level quasi-Newton optimization.	It would work not only to one modality but various more.	Provides smoother, more robust transformations in multimodal dataset.
[23]	The authors introduced the Bhattacharyya distance to target the problem of noise in joint segmentation and registration model for multimodal images.	Joint registration–segmentation variational framework with Bhattacharyya distance.	Multi level optimization.	It would work not only to one modality but various more with noise.	Provides smoother, transformations in multimodal dataset against noise as well. It can replace CV with any better boundary localization model.
[24]	The authors target the noise, intensity inhomogeneity and weak boundaries in joint segmentation and registration in multimodal images.	Unified registration–segmentation variational framework with conditional mutual information.	Multilevel discretization and Euler-Lagrange based optimization.	It is robust not only against noise but also intensity inhomogeneity and weak boundaries.	Provides robust transformations in multimodal dataset in 3D volumetric data or machine learning (ML) based prior models.

alignment of images which then refined at finer level to capture local and fine details. The common weaknesses included lack of code reproducibility and clinical auditability for translating NJSR to a practical healthcare tool or multimodality joint model is challenging due to lack of shared intensity metric like SSD. But, overall evidence supports consistency and reliability of the joint model. The SSD term here is better to use than any other like normalized cross correlation (NCC) because of its optimization stability with region based segmentation and is useful when same type of images are used like here in mono-modality. The search strategy employed both backward and forward snowballing. This method was chosen because the primary goal of this scoping review is to trace the foundational conceptual models (→ GV-JSR → NJSR) leading up to the key model (NJSR) within the chosen timeframe (2001-2024).

IV. DISCUSSION

This review found that the NJSR model consistently improves segmentation accuracy, registration smoothness, and computational efficiency relative to earlier joint models such as GV-JSR and Chan–Vese. Although precise complexity bounds are difficult to obtain, reported runtimes indicate that NJSR processed a 128 × 128 image in approximately 1.5 seconds using a multigrid solver. In comparison, the older GV-JSR model often required 3-4 seconds for a similar image size under identical computational environments.

The decrease in value mainly resulting from the linear regularization term of NJSR model that is ($S^{LC}(p)$) which solves the optimization problem. Across previous studies, NJSR model has shown elevated performance in tackling multilevel optimization. The formulated information witnessed that the NJSR gives a unified model to detect and track by making use of CV data terms with linear curvature regularization. It has resolved an issue of joint model minimization and multilevel approach. It has decreased local minimum extrema and enhanced feature of interest registration among modalities.

Many NJSR evaluations used small datasets with black backgrounds to simplify region-based segmentation. Because the energy functional computes k_1 and k_2 based on regional intensities, a non-zero background intensity simplifies calculations and improves organ delineation. However, this practice limits generalizability to clinical images with complex backgrounds. Consequently, many experiments use black backgrounds because the model relies on SSD; replacing SSD with more robust similarity measures (e.g., mutual information or gradient-based metrics) may improve generalizability to clinical tasks such as tumor tracking.

This work can be extended to 3D models or colored images. This framework can be adopted to assist clinicians in diagnosis. Where as, that is of multimodal frameworks the regmentation model is effective brain MRI, thorax and liver. It works well with clean multimodal medical dataset but is not capable of dealing with noisy medical multimodal dataset. The Badshah et al. 2022 is designed especially for noisy medical multimodal images specifically for MRI-MRI and CT-MRI cases. Its registration part performs in an efficient manner to dataset and it handles noise in a better way. Then, there is Badshah et al. 2024 model that performs far better than the models mentioned above in multimodality as it is robust against noisy images, intensity inhomogeneity, weak boundaries and complex anatomical structures. In the tables III and IV, the quantitative analysis of experiments done on the basis of metrics relative reduction (RR). Its lower values indicate the more perfect alignment between the medical images and vice-versa.

This comparative analysis of experiments is extracted from [25]. The F-measure results indicate: Experiment 1—accurate bone alignment; Experiment 2—reliable lesion delineation; Experiment 3—good clinical potential; Experiment 4—ability to handle local deformations [25]. To judge the quality of registration the relative reduction of similarity and F-measure,

$$E = D_{SSD}(M, I, P) / D_{SSD}(M, I) \quad (9)$$

where $D_{SSD}(M, I, P)$ is similarity measure after the process and $D_{SSD}(M, I)$ before the process, it measures registration improvement and with a lower value

Table 2: Comparative Analysis of Experiments for NJSR and GV-JSR Models

Experiment Number	Experiment Description	Model Name	F-measure	E-score
1	Single-object hand X-ray image	GV-JSR	0.4790	0.2343
1	Single-object hand X-ray image	NJSR	0.5200	0.0783
2	Real brain MRI image	GV-JSR	0.6839	0.2605
2	Real brain MRI image	NJSR	0.5389	0.1187
3	Global deformation using synthetic images	GV-JSR	0.7424	0.0518
3	Global deformation using synthetic images	NJSR	0.8372	0.0019
4	Local deformation using synthetic images	GV-JSR	0.3319	0.0509
4	Local deformation using synthetic images	NJSR	0.3004	0.0062

Table 3: Comparative Analysis of Experiments for [22] and [23] Models

Ex #	Description of experiments	RR values of [22]	RR values of [23]
1	Synthetic noisy images	2.3840	0.9950
2	Thorax image with intensity inhomogeneity	1.1564	0.8829
3	Synthetic medical images with intensity inhomogeneity	0.9739	0.8919

Table 4: Comparative Analysis of Experiments for [22] and [24] Models

Ex #	Description of experiments	RR values of [22]	RR values of [24]
1	Thorax	2.15547	0.907374
2	Brain (1)	0.953374	0.763422
3	Liver	1.21200	0.98055
4	Brain (2)	1.02000	0.99270

indicating better performance. For fair comparison and to avoid regridding step for the choice of parameters is such that minimum value of Jacobian matrix J of the transformation that is F must be greater than zero. This indicates that the deformed grid obtained from the displacement field is free from folds and cracks [25] and is mentioned as F-measure. The improvement is primarily in registration smoothness (E-Score), and that for certain complex real-world data (Experiment 2: Brain MRI) and local deformations (Experiment 4), a slight sacrifice in the segmentation accuracy (F-measure) is an acceptable trade-off to achieve a much smoother and more reliable deformation field (lower E-score). When the regularization parameter is very small, the underlying deformation (image transformation) has non-physical folding. The resulting transform does not maintain the grid's topology because smaller values will permit greater displacement. The determinant of the Jacobian matrix J shows this phenomenon, which is indicated by folding and cracking of the deformed grid. In the absence of a quantitative value comparison, this positive value serves to guarantee that the transformation is believable and significant. The inverse function theorem states that since the Jacobian is positive everywhere, the transformation $\phi(a)$ is locally invertible [1]. In other words, $\phi(a)$ is bijective [1].

$$J(a) = \begin{vmatrix} 1 + \partial p_1/\partial a_1 & \partial p_1/\partial a_2 \\ \partial p_2/\partial a_1 & 1 + \partial p_2/\partial a_2 \end{vmatrix} \quad (10)$$

$$F = \min(\det(J(a))) \quad (11)$$

The F-measure is a widely accepted metric for segmentation accuracy, where a value of 1.0 indicates perfect overlap. In clinical terms, F measure values above 0.8 often represent a high degree of fidelity suitable for diagnostic assessment commonly considered acceptable in many tasks. However, values below 0.5 (as seen in Experiments 2 and 4 for NJSR) suggest that the model's segmentation output may be too inaccurate for precise clinical tasks like radiotherapy planning or tumor volume tracking. Future advancements in the data fidelity term may enhance segmentation accuracy. Especially in high noise images. Therefore, improved and robust denoising can achieve better boundary detection. Thereby, enhancing F-measure efficiency while also keeping up with topological smoothness. A graphical representation of E-score [25] results is shown in Fig. 1. The experiments have been performed by the Badshah et al. 2022 model on medical or synthetic noisy images by the free available brain tumor segmentation (BRATS) 2015 dataset by adding different noise levels. To assess the registration the Bhattacharyya (BC) distance is determined as follows [23]:

$$RR_{BC} = D_{BC}(M, I, P) / D_{BC}(M, I) \quad (12)$$

where $D_{BC}(M, I, P)$ is the Bhattacharyya distance term after registration of the transformed moving image and target image, and $D_{BC}(M, I)$ represents the BC distance term before registration. In first experiment between [22], [23], the models were compared on a pair of noisy synthetic images with a resolution of 128 × 128. The second experiment was performed on pair of thorax MR medical dataset with intensity inhomogeneity with 128 × 128 resolution. The third experiment, incorporates synthetic medical images with intensity inhomogeneity on 128 × 128 resolution. A graphical representation of relative reduction (RR) [23] results is shown in Fig. 2.

Similarly, the Badshah et al. 2024 model has been validated on images having intensity inhomogeneity, large deformations and high level of noise. The metric relative reduction of similarity measure is defined as follows and is used for quantitative comparison to judge the registration quality for multi-modality images [24].

$$RR_{CMI} = D_{CMI}(M, I, P) / D_{CMI}(M, I) \quad (13)$$

where $D_{CMI}(M, I, P)$ is the conditional mutual information distance measure term after registration of the transformed moving image and target image, and $D_{CMI}(M, I)$ shows the conditional mutual information (CMI) distance measure before registration. The experiments were performed on different multi modal medical images that are thorax, brain and liver of size 128 × 128. A graphical representation of RR [22], [24] results is shown in Fig. 3.

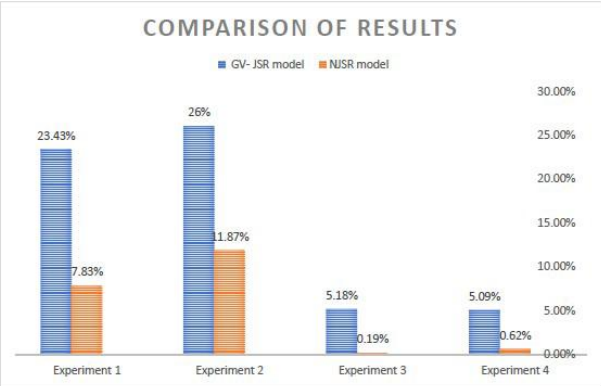

Figure 1. E-score expressed as a percentage comparing [25] and [17] models and lower bars indicate less differences and better performance.

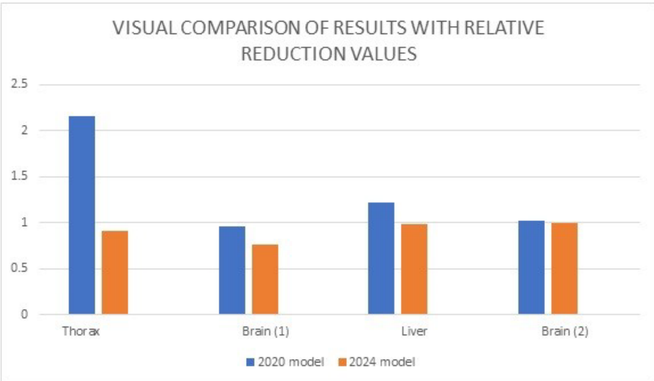

Figure 2. RR performance achieved by [22] and [23] models across different datasets and higher bars indicate better performance.

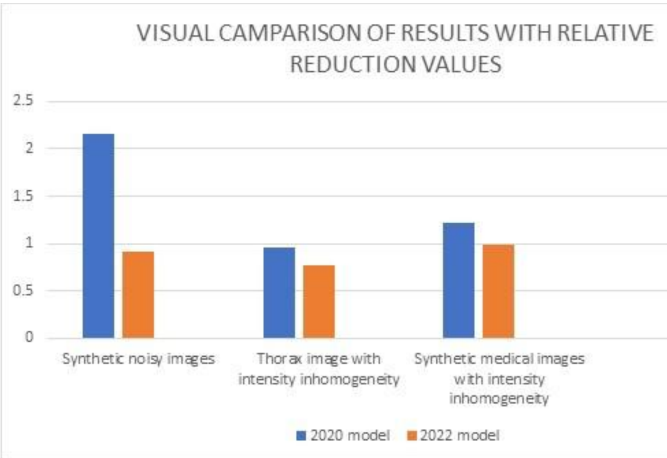

Figure 3. RR performance achieved by [22] and [24] models across different datasets and higher bars indicate better performance.

V. CONCLUSION

This review demonstrates that the NJSR model significantly improves upon earlier methods. Moreover, it is also improved and utilized in multi modal frameworks. It employs SSD metrics and methods such as B-spline deformation fields, PDE solvers, and quasi-Newton optimization. The transition from nonlinear elasticity regularizer (GV-JSR) to linear curvature regularizer (NJSR) has efficiently improved the convergence and registration smoothness (reduced E-score) by solving an optimization problem more effectively. The NJSR model, utilizing linear curvature smoothing, shown an efficient reduction in processing time (e.g., approximately 1.5 seconds vs. 3–4 seconds for GV-JSR on a 128 × 128 image) and achieved outstanding registration improvement (lower E-score) across all experiments. Future work should include hyperparameter optimization techniques (e.g., Bayesian optimization) to automate tuning of regularization weights. Although efforts have been made to select parameters for registration, as mentioned by Larrey Ruiz et al. to choose the best parameters for the regularization term in the joint segmentation and registration model, more effort is needed. It is also recommended to extend the framework to larger 3-D datasets, and clinical validation.

The notion of selective segmentation can be extended to complex 3D medical datasets. It would enable accurate performance to process large resolution clinical images. Moreover, preserving medically focused topological structures. Refinements such as selective segmentation—focusing computational effort on regions of interest (e.g., tumors)—may overcome the black-background limitation. Testing NJSR on large, public, and highly deformable datasets is recommended to establish generalizability. Additionally, extending NJSR and other multi modal frameworks explored by Rada et al. to vector-valued (color) images—using appropriate data-fidelity terms and color distance metrics (e.g., L2 norm)—is a promising direction. Finally, variational frameworks remain relevant in the deep learning era. The mathematical energy functional from models like NJSR, regmentation, Badshah et al. 2022 and Badshah et al. 2024 models which enforces geometric constraints (like smoothness and nonfolding via the Jacobian determinant), can be directly adapted and used as a robust, mathematically-principled loss function (or regularization term) within modern deep learning (DL) architectures demonstrated by Rada et al. for joint segmentation and registration. This hybrid approach ensures that the high speed of deep learning (DL) is balanced by the physical realism and fidelity of classical variational mathematics.

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